Low Constant Tangential Thrust Spiral Trajectories

Chong-Hung Zee*
Curtiss-Wright Corporation, Wood-Ridge, N. J.

The stroboscopic method in nonlinear mechanics is applied to the calculations of a low constant tangential thrust spiral trajectory in a single force field. Closed-form solutions for the first approximation are derived. The oscillatory nature of the spiral is exposed; the maximum deviation of the oscillation is of the order of $2/A$, where $1/A$ is the initial thrust to local weight ratio, and the oscillations damp out as the spiral proceeds.

Nomenclature

A = $M_0 g_0/\dot{m}\bar{c}$
$\bar{c}$ = average exhaust velocity of jet
g_0 = gravitational acceleration at $r = r_0$
K = integration constant
$\dot{m}$ = constant mass flow rate
M_0 = mass of space vehicle at beginning of thrusting
p = $r^2\theta/r_0^2\omega = \dot{\theta}/u^2\omega$
r = distance between space vehicle and center of attraction
S = defined by Eq. (9)
t = time
T = stroboscopic time
ΔT = $2\pi/A$ = increment in stroboscopic time
u = r_0/r
v = velocity of space vehicle
θ = polar angle of any point on trajectory
τ = $(g_0/\bar{c})t$
ϕ = angle between tangent line and the normal of radius r
Φ = integration constant
ψ = defined by Eq. (9)
ω = angular velocity of space vehicle at $t = 0$

Subscripts

0 = value at $t = 0$ or value of zero-order approximation
1 = value of first-order approximation
n = value designated at any particular point

Introduction

In a previous paper,[1] low-thrust oscillatory spiral trajectories under a constant thrust acceleration were discussed which have the differential equations of the autonomous type in nonlinear mechanics,[2] because the thrust acceleration remains constant during the flight. For the case of constant thrust, the thrust acceleration varies with time because of the consumption of the fuel; thus the differential equations become the nonautonomous type, and the concept of stroboscopic method[3] will be employed in solving the present problem.

Based on the energy relationships, Melbourne[4] derived the equation of a spiral trajectory under a constant tangential thrust commencing from a circular orbit, which may be conceived as a mean path of the spiral. This paper deals with the same problem by solving the equations of motion directly, and the oscillatory effect of the trajectory will be brought out.

Analysis

The equations of motion for a space vehicle under the influence of a low constant tangential thrust and a single center of attraction are (see Fig. 1)

$$\ddot{r} - r\dot{\theta}^2 = -\frac{g_0 r_0^2}{r^2} + \frac{\dot{m}\bar{c}}{M_0 - \dot{m}t}\sin\phi \tag{1}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{\dot{m}\bar{c}}{M_0 - \dot{m}t}\cos\phi \tag{2}$$

and, by definition,

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \qquad \sin\phi = \dot{r}/v \qquad \cos\phi = r\dot{\theta}/v$$

Following the usual procedures in eliminating dt and letting

$$u = r_0/r \qquad p = r^2\dot{\theta}/r_0^2\omega = \dot{\theta}/u^2\omega \tag{3}$$

Eqs. (1) and (2) can be reduced to

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{p^2} \tag{4}$$

$$\frac{dp}{d\theta} = \frac{\dot{m}\bar{c}}{(M_0 - \dot{m}t)g_0}\cdot\frac{1}{u^2p}\cdot\frac{1}{[u^2 + (du/d\theta)^2]^{1/2}} \tag{5}$$

Setting

$$\tau = t \times (g_0/\bar{c}) \qquad A = M_0 g_0/\dot{m}\bar{c} \tag{6}$$

Eq. (5) is simplified as

$$\frac{dp}{d\theta} = \frac{1}{A - \tau}\cdot\frac{1}{u^2p}\cdot\frac{1}{[u^2 + (du/d\theta)^2]^{1/2}} \tag{7}$$

Equations (4) and (7) are to be solved in accordance with the stroboscopic method.

In view of the smallness of $1/(A - \tau)$ ($\ll 1$), the approximation of the zero order of Eq. (7) is

$$dp/d\theta \cong 0 \qquad p = p_0 \text{ (arbitrary constant)} \tag{8}$$

Hence the solution (zero-order approximation) of Eq. (4) is

$$u = (1/p_0^2) + S_0\cos(\theta + \psi_0) \tag{9}$$

where S_0 and ψ_0 are integration constants. And, accordingly,

$$du/d\theta = -S_0\sin(\theta + \psi_0) \tag{10}$$

Equation (9) is an ellipse, which has an eccentricity of $S_0 p_0^2$.

For a space vehicle commencing from a circular orbit, the trajectory of the vehicle remains nearly circular; hence $S_0 p_0^2$ is very small. From Eqs. (3, 6, 8, and 9), one has

$$\theta = (1/p_0^3)\omega(\bar{c}/g_0)\tau \tag{11}$$

as a zero-order approximation for τ by omitting the terms containing $S_0 p_0^2$ and $(S_0 p_0^2)^2$.

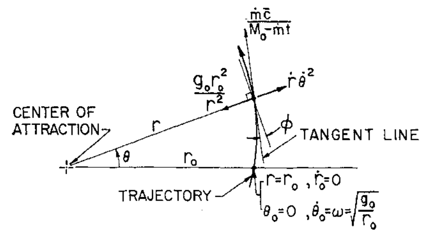

Fig. 1 Notations

Received by ARS December 3, 1962; revision received April 17, 1963.

* Project Engineer, Wright Aeronautical Division; now Dynamicist, Dynamic Analysis Advanced Development Group, Grumman Aircraft Engineering Corporation, Bethpage, N. Y.

Substituting Eqs. (8–11) into Eq. (7), the resulting equation is

$$\frac{dp}{d\theta} = \frac{p_0^5}{A - (g_0p_0^3/\bar{c}\omega)\theta}[1 + S_0p_0^2\cos(\theta + \psi_0)]^{-2}\{[1 + S_0p_0^2\cos(\theta + \psi_0)]^2 + [S_0p_0^2\sin(\theta + \psi_0)]^2\}^{-1/2} \quad (12)$$

After expanding the brackets and braces on the right-hand side into series and omitting the terms containing $(S_0p_0^2)^2$ or higher orders, Eq. (12) becomes

$$\frac{dp_1}{d\theta} = \frac{p_0^5}{A - (g_0p_0^3/\bar{c}\omega)\theta}[1 - 3S_0p_0^2\cos(\theta + \psi_0)] \quad (13)$$

where a subscript 1 is added to p on the left-hand side, and p_1 may be conceived as the first approximation of p. In this paper, only the first approximation is investigated; hence, in the following text, all p's without subscript 1 are actually p_1's.

Based on the stroboscopic method, the increment of p during a period of 2π in θ is

$$\Delta p = \int_{\theta_n}^{\theta_n+2\pi}\frac{p_0^5}{A - (g_0p_0^3/\bar{c}\omega)\theta}[1 - 3S_0p_0^2\cos(\theta + \psi_0)]d\theta \quad (14)$$

Upon examining the factor

$$\frac{1}{A - (g_0p_0^3/\bar{c}\omega)\theta}\left(= \frac{1}{A - \tau} \ll 1\right)$$

if it is expanded in a Taylor's series at $\theta = \theta_n$,

$$\frac{1}{A - (g_0p_0^3/\bar{c}\omega)\theta} = \frac{1}{A - (g_0p_0^3/\bar{c}\omega)\theta_n} + \frac{(g_0p_0^3/\bar{c}\omega)}{[A - (g_0p_0^3/\bar{c}\omega)\theta_n]^2}\cdot\Delta\theta + \ldots \quad (15)$$

The second term on the right-hand side, being of smaller order of magnitude, even when $\Delta\theta$ has a maximum value of 2π, can be dropped out; hence Eq. (14) may be written as

$$\Delta p = \frac{p_0^5}{A - (g_0p_0^3/\bar{c}\omega)\theta_n}\int_{\theta_n}^{\theta_n+2\pi}[1 - 3S_0p_0^2\cos(\theta + \psi_0)]d\theta = \frac{2\pi}{A}\cdot\frac{p_0^5}{1 - (g_0p_0^3/A\bar{c}\omega)\theta_n} \quad (16)$$

Setting $2\pi/A = \Delta T$ (T is called the stroboscopic time) and dropping out the subscript n on θ, Eq. (16) yields

$$\frac{\Delta p}{\Delta T} = \frac{p_0^5}{1 - (g_0p_0^3/A\bar{c}\omega)\theta} = \frac{p_0^5}{1 - (1/A)\tau} \quad (17)$$

By introducing a passage to the limit in the sense of the stroboscopic method, Eq. (17) becomes

$$dp/dT = p^5/[1 - (1/A)\tau] \quad (18)$$

which is known as the stroboscopic differential equation.

In order to solve Eq. (18), the relationship between T and τ should be examined. Returning to Eq. (17) and considering the fact that, for the same increment of $\Delta p, \Delta\theta$ is 2π and ΔT increases $2\pi/A$, the quasi-continuous relationship between T and θ can be established as

$$\Delta T = (1/A)\Delta\theta \quad (19)$$

From Eq. (11), one has

$$\Delta\theta = (\omega/p_0^3)\cdot(\bar{c}/g_0)\Delta\tau \quad (20)$$

Substituting Eqs. (19) and (20) into Eq. (17) yields

$$\Delta p/\Delta\tau = [p_0^2\omega/(A - \tau)]\cdot(\bar{c}/g_0) \quad (21)$$

and

$$dp/d\tau = [p^2\omega/(A - \tau)]\cdot(\bar{c}/g_0) \quad (22)$$

which has a solution

$$1/p = (\bar{c}/g_0)\omega\ln[1 - (\tau/A)] + 1 \quad (23)$$

with the aid of the initial condition $p = 1, \tau = 0$.

In Eqs. (9) and (10), p_0, S_0, and ψ_0 are considered as constants for the zero-order approximation of the problem. In order to obtain the first approximation, better approximations are required for p, S, and ψ, where the first approximation of p already is given by Eq. (23). The first approximation of S and ψ will be examined by considering them as functions of θ. Differentiating Eqs. (9) and (10) with respect to θ and with the aid of Eq. (4) yields

$$\frac{dS_1}{d\theta} = \frac{2}{p_0^3}\cos(\theta + \psi_0)\frac{dp_1}{d\theta} \quad (24)$$

$$\frac{d\psi_1}{d\theta} = -\frac{2\sin(\theta + \psi_0)}{p_0^3S_0}\cdot\frac{dp_1}{d\theta} \quad (25)$$

where $dp_1/d\theta$ is given by Eq. (13).

By employing the same argument used in obtaining Δp in Eq. (16), the increments of ΔS_1 and $\Delta\psi_1$ during a period of 2π in θ are

$$\Delta S_1 = -3S_0p_0^4\cdot\frac{1}{1 - (g_0p_0^3/A\bar{c}\omega)\theta_n}\cdot\frac{2\pi}{A} \quad (26)$$

$$\Delta\psi_1 = 0 \quad (27)$$

and the corresponding stroboscopic differential equations are

$$dS/dT = -3Sp^4\cdot\{1/[1 - (1/A)\tau]\} \quad (28)$$

$$d\psi/dT = 0 \quad (29)$$

Combining Eqs. (18) and (28) yields

$$dp/dS = -p/3S \quad (30)$$

which has a solution

$$KS = 1/p^3 \quad (31)$$

where K is an integration constant.

Equation (29) gives

$$\psi = \Phi \quad (32)$$

where Φ is another integration constant.

With the expressions of the first approximations of S [Eq. (31)] and ψ [Eq. (32)], Eq. (9) becomes

$$u = (1/p^2) + (1/Kp^3)\cos(\theta + \Phi) \quad (33)$$

By the assumption that the space vehicle is initially in a circular orbit, the initial conditions are

$$u = 1 \qquad du/d\theta = 0 \qquad p = 1 \qquad \text{at } t = 0, \theta = 0$$

Hence, from Eqs. (9, 10, and 13), one has

$$(dp/d\theta)_0 = 1/A \quad (34)$$

and

$$\Phi = \pi/2 \qquad K = -A/2 \quad (35)$$

are obtained by Eqs. (33) and (34) with the initial conditions; thus Eq. (33) has the form

$$u = (1/p^2) + (2/Ap^3)\sin\theta \quad (36)$$

The first approximation between θ and τ is obtained by replacing p_0 as p_1 in Eq. (11):

$$\theta = (1/p^3)\cdot(\omega\bar{c}/g_0)\tau \quad (37)$$

Equations (23, 36, and 37) describe the desired spiral trajectory. As is seen from Eq. (36), the spiral trajectory is of oscillatory nature, which damps out as the trajectory progresses. If the second term in Eq. (36) is dropped out because of $A \gg 1$, Eq. (36) yields

$$u = 1/p^2 \tag{38}$$

In combining with Eq. (23), Eq. (38) becomes

$$u = \{\omega(\bar{c}/g_0)\ln[1 - (\tau/A)] + 1\}^2 \tag{39}$$

which is the result obtained by Melbourne.[4]

Conclusions

Briefly, application of the stroboscopic method in nonlinear mechanics has been shown to provide concise solutions to the spiral trajectory under low constant tangential thrust previously discussed in the literature. The oscillatory nature of the spiral is brought out. The oscillations cause a maximum deviation from the mean path of the order of $2/A$ during the early stage of spiraling and gradually damp out as spiral proceeds. Figure 2 shows such an oscillatory spiral trajectory for a space vehicle ($\bar{c}$ = 161,000 fps), which is initially in a 300-naut-mile orbit around the earth and is subject to a constant tangential thrusting force of $0.005M_0g_0$.

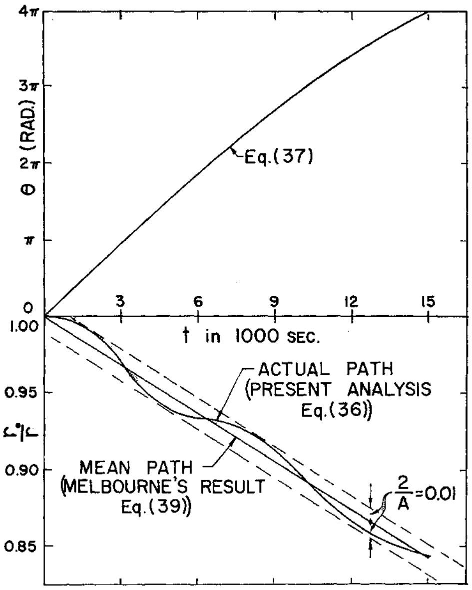

Fig. 2 Oscillatory spiral trajectory

References

[1] Zee, C.-H., "Low-thrust oscillatory spiral trajectory," Wright Aeronaut. Div., Curtiss-Wright Corp., Wood-Ridge, N. J , IOM 102462 (October 1962).

[2] Minorsky, N., *Nonlinear Oscillations* (D. Van Nostrand Co. Inc., Princeton, N. J., 1962), Chap. 14, pp. 329–335.

[3] Minorsky, N., *Nonlinear Oscillations* (D. Van Nostrand Co. Inc., Princeton, N. J., 1962), Chap. 16, pp. 390–415.

[4] Melbourne, W. G., "Interplanetary trajectories and payload capabilities of advance propulsion vehicles," Jet Propulsion Lab., Calif. Inst. Tech., Pasadena, Calif., TR 32-68 (March 1961).

JULY 1963 AIAA JOURNAL VOL. 1, NO. 7

Stability Boundaries of Liquid-Propelled Space Vehicles with Sloshing

HELMUT F. BAUER*
NASA George C. Marshall Space Flight Center, Huntsville, Ala.

For liquid-propelled space vehicles with large diameter propellant containers, the effects of propellant sloshing upon the vehicle stability are becoming more critical, especially since at launch a very large amount of the total weight is in form of liquid propellant. Describing the propellant motion with a mechanical model, the influence of propellant sloshing in one, two, and three tanks is determined with the Hurwitz stability boundaries. They are represented as the necessary required damping in the containers vs the container location. The influence of the various parameters, such as control frequency and control damping of the vehicle, as well as sloshing mass, sloshing frequency, and tank locations, is investigated. For a space vehicle in which the propellant of only one tank is free to oscillate, the danger zone where baffles may have to be applied to maintain stability is located between the center of gravity and the center of instantaneous rotation. The influence of tank geometry such as concentric containers or a quarter-tank arrangement is investigated also. For a tandem arrangement of two and three containers in which the propellant is free to oscillate, the danger zone shifts with increasing distance between the slosh masses toward the rear of the space vehicle, indicating in a practical case that most of the booster tanks of a space vehicle have to be provided with appropriate baffles to maintain vehicle stability.

I. Introduction

THE motion of the liquid propellant in the tanks of a space vehicle represents, due to its low natural frequencies that are usually very close to the control frequency, a potential hazard for stability and control. The stability of a space vehicle can be influenced tremendously by this propellant sloshing. Very useful results can be obtained from simplified stability investigations of a rigid space vehicle that is controlled by a simple control system. By variation of parameters, the possibility of the destabilizing effect of the propellant is investigated. The propellant motion in the containers will be described by the mechanical model as derived in Ref. 1. The stability boundaries are given by the necessary damping values of the propellant along the vehicle.

Received March 10, 1963.

* Formerly Chief of Flutter and Vibration Section, Dynamics Branch, Aeroballistic Division; now Professor of Engineering Mechanics, School of Engineering Mechanics, Georgia Institute of Technology.